Suggested solutions to the (resit) Contract Theory exam on February 19, 2014
VERSION: March 13, 2014

Question 1 (adverse selection)

Consider the following model of a market for pencils that can be produced in different qualities. There are a continuum of consumers (the "agent" of the adverse selection model), each of whom purchasing either one pencil or no pencil. A fraction $\nu \in (0, 1)$ of the consumers have a high valuation for pencil quality and the remaining fraction $(1 - \nu)$ have a low valuation for pencil quality (and the total number of consumers is normalized to one). The high-valuation consumers' payoff if consuming one pencil of quality $\overline{q}$ at the price $\bar{t}$ is given by

$$\overline{\theta}\overline{q} - \bar{t},$$

where $\overline{\theta} > 0$ is a parameter. The low-valuation consumers' payoff if consuming one pencil of quality $\underline{q}$ at the price $\underline{t}$ is given by

$$\underline{\theta}\underline{q} - \underline{t},$$

where $\underline{\theta}$ is a parameter satisfying $\overline{\theta} > \underline{\theta} > 0$. If the consumers (both the high- and low-valuation ones) choose not to consume any pencil at all, their payoff is zero. There is a firm (the "principal" of the adverse selection model) that has a monopoly in the pencil market. If selling one pencil of quality $\underline{q}$ to each of the low-valuation consumers and one pencil of quality $\overline{q}$ to each of the high-valuation consumers, the firm incurs the production costs

$$\frac{1 - \nu}{2}\underline{q}^2 + \frac{\nu}{2}\overline{q}^2.$$

The firm's total profits are therefore given by

$$(1 - \nu)\,\underline{t} + \nu\bar{t} - \frac{1 - \nu}{2}\underline{q}^2 - \frac{\nu}{2}\overline{q}^2.$$

Each consumer knows his or her own θ perfectly. However, the monopoly firm does not know the θ of an individual consumer, but only that a fraction ν of the consumers have a high valuation and that the rest have a low valuation. The objective of the firm is to maximize its total profits.

a) **Suppose the parameters are such that the firm optimally interacts with both kinds of consumers. Formulate the optimization problem that the firm faces when designing the menu of prices: state the objective function and the constraints, and explain what the choice variables are. Explain the meaning of the constraints in words.**

- The firm's objective function is given by its total profits:

$$(1-\nu)\,\underline{t} + \nu\bar{t} - \frac{1-\nu}{2}\underline{q}^2 - \frac{\nu}{2}\overline{q}^2.$$

  The firm wants to maximize that expression with respect to the choice variables $\underline{t}$, $\bar{t}$, $\underline{q}$, and $\overline{q}$, subject to the following four constraints:

  – The low-valuation customers must prefer their bundle to no bundle at all (individual rationality for the L-type):

$$\underline{\theta}\underline{q} - \underline{t} \geq 0. \tag{IR-L}$$

  – The high-valuation customers must prefer their bundle to no bundle at all (individual rationality for the H-type):

$$\overline{\theta}\overline{q} - \bar{t} \geq 0. \tag{IR-H}$$

  – The low-valuation customers must prefer their bundle to the high-valuation customers' bundle (incentive compatibility for the L-type):

$$\underline{\theta}\underline{q} - \underline{t} \geq \underline{\theta}\overline{q} - \bar{t}. \tag{IC-L}$$

  – The high-valuation customers must prefer their bundle to the low-valuation customers' bundle (incentive compatibility for the H-type):

$$\overline{\theta}\overline{q} - \bar{t} \geq \overline{\theta}\underline{q} - \underline{t}. \tag{IC-H}$$

b) **Prove formally that any pair of qualities $\left(\underline{q}, \overline{q}\right)$ that satisfy the constraints under a) also satisfy $\underline{q} \leq \overline{q}$.**

- To prove this we need only two of the four constraints, namely (IC-L) and (IC-H). Adding these constraints yields

$$\left(\underline{\theta}\underline{q} - \underline{t}\right) + \left(\overline{\theta}\overline{q} - \bar{t}\right) \geq \left(\underline{\theta}\overline{q} - \bar{t}\right) + \left(\overline{\theta}\underline{q} - \underline{t}\right)$$

  or, since the transfers cancel out,

$$\underline{\theta}\underline{q} + \overline{\theta}\overline{q} \geq \underline{\theta}\overline{q} + \overline{\theta}\underline{q}.$$

  Rewriting this, we have

$$\left(\overline{\theta} - \underline{\theta}\right)\left(\overline{q} - \underline{q}\right) \geq 0.$$

  But since $\overline{\theta} - \underline{\theta} > 0$ by assumption, the last inequality simplifies to $\overline{q} - \underline{q} \geq 0$, which we were supposed to prove. That is, the two incentive compatibility constraints imply monotonicity ($\underline{q} \leq \overline{q}$). More generally, we know from the course that in adverse selection models monotonicity is implied by the IC constraints and the Spence-Mirrlees (or single-crossing) condition. Here, however, the Spence-Mirrlees condition is implicit in our chosen functional forms.

c) **Let the first-best levels of $\underline{q}$ and $\overline{q}$ be defined as the ones that maximize the total surplus,**

$$(1-\nu)\,\underline{\theta}\underline{q}+\nu\overline{\theta}\overline{q}-\frac{1-\nu}{2}\underline{q}^2-\frac{\nu}{2}\overline{q}^2.$$

Calculate these first-best levels. Explain the economic intuition behind your result.

- It is stated in the question that the first best levels are defined as the ones that maximize the above expression for the total surplus. To calculate these we can take the first-order conditions with respect to $\underline{q}$ and $\overline{q}$. Doing that yields

  $$\frac{\partial}{\partial \underline{q}}\left[(1-\nu)\,\underline{\theta}\underline{q}+\nu\overline{\theta}\overline{q}-\frac{1-\nu}{2}\underline{q}^2-\frac{\nu}{2}\overline{q}^2\right]=(1-\nu)\,\underline{\theta}-(1-\nu)\,\underline{q}=0\Rightarrow \underline{q}^{FB}=\underline{\theta}$$

  and

  $$\frac{\partial}{\partial \overline{q}}\left[(1-\nu)\,\underline{\theta}\underline{q}+\nu\overline{\theta}\overline{q}-\frac{1-\nu}{2}\underline{q}^2-\frac{\nu}{2}\overline{q}^2\right]=\nu\overline{\theta}-\nu\overline{q}=0\Rightarrow \overline{q}^{FB}=\overline{\theta}.$$

  (The second-order condition is clearly satisfied as the objective is quadratic in the choice variables and the coefficients for the quadratic terms are negative.)
- The intuition is that these quantities are the ones that, given a known value of θ, ensure that the agent's marginal benefit of consuming the good (MB) is equal to the principal's marginal cost of producing the good (MC). If we had MB$\neq$MC, total surplus wouldn't be maximized.

d) **Now return to the second-best problem you have formulated under a). Solve this problem. Explain how the optimal second-best qualities differ from the optimal first-best qualities. Also explain the economic intuition behind any differences. Which type, if any, gets any rents at the second-best optimum? Why?**

- We can solve the problem by making use of the following five-step recipe:
  1 Show that IR-L and IC-H imply IR-H, so we can ignore IR-H.
  2 *Guess* that IC-L doesn't bind.
  3 Inspect the problem and note that the two remaining constraints must bind. Therefore we can plug them into the objective function.
  4 Solve the resulting unconstrained problem.
  5 Verify that the solution satisfies IC-L (i.e., that the guess at (2) was correct).
- The claim that IR-L and IC-H imply IR-H can be proven as follows:

  $$\overline{\theta}\overline{q}-\overline{t}\geq\overline{\theta}\underline{q}-\underline{t}>\underline{\theta}\underline{q}-\underline{t}\geq 0. \qquad (1)$$

  The first inequality is the same as IC-H. The second inequality follows from the assumption that $\overline{\theta}>\underline{\theta}$ (and the fact that $\underline{q}>0$). The third inequality is the same as IR-L. The above sequence of inequalities means that $\overline{\theta}\overline{q}-\overline{t}\geq 0$, which is the same as IR-H, so we have proven the claim.

- If we also guess that IC-L doesn't bind, the remaining constraints are IR-L and IC-H:

$$\underline{\theta}\underline{q}-\underline{t}\geq 0, \tag{IR-L}$$

$$\overline{\theta}\overline{q}-\overline{t}\geq \overline{\theta}\underline{q}-\underline{t}. \tag{IC-H}$$

- The objective is increasing in $\underline{t}$ and $\overline{t}$. Therefore if one or both of the constraints did not bind, the principal would be able to increase his payoff. That is, the two constraints must both bind at the optimum.
- Setting the constraints to equality and solving for $\underline{t}$ and $\overline{t}$ yield

$$\underline{t}=\underline{\theta}\underline{q}, \tag{2}$$

$$\begin{aligned}\overline{t} &= \overline{\theta}\overline{q}-\overline{\theta}\underline{q}+\underline{t}\\ &= \overline{\theta}\overline{q}-\left(\overline{\theta}-\underline{\theta}\right)\underline{q} \end{aligned}\tag{3}$$

- Plugging into the objective:

$$\begin{aligned}\pi &= (1-\nu)\,\underline{t}+\nu\overline{t}-\frac{1-\nu}{2}\underline{q}^2-\frac{\nu}{2}\overline{q}^2\\ &= (1-\nu)\,\underline{\theta}\underline{q}+\nu\left[\overline{\theta}\overline{q}-\left(\overline{\theta}-\underline{\theta}\right)\underline{q}\right]-\frac{1-\nu}{2}\underline{q}^2-\frac{\nu}{2}\overline{q}^2\end{aligned}$$

- The first-order conditions:

$$\frac{\partial\pi}{\partial\underline{q}}=(1-\nu)\,\underline{\theta}-\nu\left(\overline{\theta}-\underline{\theta}\right)-(1-\nu)\,\underline{q}=0\Rightarrow \underline{q}^{SB}=\underline{\theta}-\frac{\nu\left(\overline{\theta}-\underline{\theta}\right)}{1-\nu} \tag{4}$$

$$\frac{\partial\pi}{\partial\overline{q}}=\nu\overline{\theta}-\nu\overline{q}=0\Rightarrow\overline{q}^{SB}=\overline{\theta}. \tag{5}$$

- We also need to show that IC-L is satisfied at the (possible) solution we have found:

$$\underline{\theta}\underline{q}^{SB}-\underline{t}^{SB}\geq\underline{\theta}\overline{q}^{SB}-\overline{t}^{SB} \tag{IC-L}$$

or (using (2) and (3))

$$\underline{\theta}\underline{q}^{SB}-\underline{\theta}\underline{q}^{SB}\geq\underline{\theta}\overline{q}^{SB}-\left[\overline{\theta}\overline{q}^{SB}-\left(\overline{\theta}-\underline{\theta}\right)\underline{q}^{SB}\right] \tag{IC-L}$$

or

$$\left(\overline{\theta}-\underline{\theta}\right)\left(\overline{q}^{SB}-\underline{q}^{SB}\right)\geq 0 \tag{IC-L}$$

or (because $\overline{\theta}-\underline{\theta}>0$)

$$\overline{q}^{SB}\geq\underline{q}^{SB} \tag{IC-L}$$

or (using (4) and (5))

$$\overline{\theta}\geq\underline{\theta}-\frac{\nu\left(\overline{\theta}-\underline{\theta}\right)}{1-\nu}, \tag{IC-L}$$

which clearly is satisfied (as $\overline{\theta}>\underline{\theta}$). We conclude that since IC-L is satisfied at the (possible) solution, this is indeed the solution.

- We thus have $\overline{q}^{SB} = \overline{q}^{FB}$ (efficiency/no distortion at the top) and $\underline{q}^{SB} < \underline{q}^{FB}$ (inefficiency/distortion at the bottom).
- We also conclude that the L-type does not get any rents (i.e., any utility on top of what that agent gets for his outside option), as IR-L binds. However, IR-H is satisfied with a strict inequality at the optimum — this follows already from (1). So we have rent extraction at the bottom but not at the top.
- **Intuition:** Key to the results is that the high type is the one who gets, for any given q, both: (i) the highest *marginal* utility [the "single-crossing condition"] and (ii) the highest *total* utility.
- Because of (ii), the firm primarily wants to extract the high type's surplus (as it's larger).
  - However, if the high type gets too little, he can choose the low type's bundle instead.
- To prevent this, the monopolist makes the low type's bundle less attractive by offering those consumers less.
- This works because of (i): The high type suffers more from a reduction in q than the low type.

Question 2 (moral hazard)

Prometheus Sørensen (the principal, P for short) owns a factory producing pencils and wants to hire Absalon Nielsen (the agent, A for short) to work there. If hired, A's task will be to operate a pencil machine and to make sure it runs smoothly. To do this well, A must "make an effort", which involves a (personal) cost to A. This is modelled as A's choosing an effort level $e \in \{0,1\}$, where $e = 1$ means "making an effort" and $e = 0$ means "not making an effort". The associated cost equals

$$\psi(e) = \begin{cases} \psi & \text{if } e = 1 \\ 0 & \text{if } e = 0, \end{cases}$$

with $\psi > 0$. The number of pencils that come out of the machine, q, is either large ($q = \overline{q}$) or small ($q = \underline{q}$), with $\overline{q} > \underline{q} > 0$. The probability that the number is large depends on whether A has made an effort or not:

$$\Pr(q = \overline{q} \mid e) = \begin{cases} \pi_1 & \text{if } e = 1 \\ \pi_0 & \text{if } e = 0, \end{cases}$$

with $0 < \pi_0 < \pi_1 < 1$. P (and the court) can observe which quantity that is realized ($\overline{q}$ or $\underline{q}$) but not the effort level chosen by A. It is assumed that P has all the bargaining power and makes a take-it-or-leave-it offer to A. A contract can specify two numbers, $\overline{t}$ and $\underline{t}$, where $\overline{t}$ is the payment to A if $q = \overline{q}$, and $\underline{t}$ is the payment to A if $q = \underline{q}$. P is risk neutral and his payoff, given a quantity q and a payment t, equals

$$V = q - t.$$

A is also risk neutral and his payoff, given a payment t and an effort level e, equals

$$U = t - \psi(e).$$

A is protected by limited liability, meaning that $\overline{t} \geq 0$ and $\underline{t} \geq 0$. A's outside option would yield the payoff $\widehat{U} \geq 0$.

a) **Assume that $\widehat{U} = 0$. Calculate (analytically, not using a figure) P's cost of implementing the high effort level when (i) P can observe A's effort (i.e., the first best) and (ii) when P cannot observe A's effort (i.e., the second best). Compare these costs and explain in what sense effort is underprovided in the model due to asymmetric information.**

- To implement a high effort when the effort is observable will cost

$$C^{FB} = \psi.$$

This is the cost that the agent himself incurs when making a high effort. The principal can write into the contract that the agent must exert a high effort (as the effort is observable), but as compensation the principal must pay at least ψ for the agent to accept the contract (as the outside option gives utility zero). However, the principal does not need to pay more than that (if being paid ψ the agent is indifferent between accepting and rejecting, and so there is an equilibrium in which he does accept — the convention in the contract theory literature is to focus on that equilibrium).

- To implement a high effort when effort is not observable, the principal should solve the following problem:

$$\max_{\bar{t},\underline{t}} \left\{\pi_1\left(\bar{q}-\bar{t}\right)+(1-\pi_1)\left(\underline{q}-\underline{t}\right)\right\} \text{ subject to}$$

$$\pi_1\bar{t}+(1-\pi_1)\,\underline{t}-\psi \geq \pi_0\bar{t}+(1-\pi_0)\,\underline{t} \Leftrightarrow (\pi_1-\pi_0)\left(\bar{t}-\underline{t}\right)-\psi \geq 0 \tag{IC}$$

$$\pi_1\bar{t}+(1-\pi_1)\,\underline{t}-\psi \geq 0, \tag{IR-H}$$

$$\underline{t} \geq 0 \qquad \text{and} \qquad \bar{t} \geq 0. \tag{LL-L and LL-H}$$

- Since $\psi > 0$ and $\pi_1 - \pi_0 > 0$, IC implies that $\bar{t} > \underline{t}$, which in turn means that LL-H must be lax.
- Moreover, IC and the two LL-constraints imply IR-H, so we can ignore IR-H.
- The Lagrangian:

$$\mathcal{L} = \pi_1\left(\overline{S}-\bar{t}\right)+(1-\pi_1)\left(\underline{S}-\underline{t}\right)+\lambda\left[(\pi_1-\pi_0)\left(\bar{t}-\underline{t}\right)-\psi\right]+\xi\underline{t}$$

- FOC w.r.t. $\bar{t}$:

$$\frac{\partial\mathcal{L}}{\partial\bar{t}} = -\pi_1+\lambda\,(\pi_1-\pi_0) = 0,$$

which immediately shows that IC binds as $\lambda > 0$.

- FOC w.r.t. $\underline{t}$:

$$\frac{\partial\mathcal{L}}{\partial\underline{t}} = -\,(1-\pi_1)-\lambda\,(\pi_1-\pi_0)+\xi = 0$$

- Adding up the two FOCs yields

$$\xi = 1, \tag{6}$$

which means that LL-L must be binding.

- We thus know that IC and LL-L bind. The latter means that

$$\underline{t}^{SB} = 0,$$

and plugging that expression for $\underline{t}^{SB}$ into the binding IC yields

$$\bar{t}^{SB} = \frac{\psi}{\pi_1-\pi_0}.$$

- The cost of implementing the high effort level when effort is not observable is thus

$$C^{SB} = \pi_1\bar{t}^{SB}+(1-\pi_1)\,\underline{t}^{SB} = \frac{\pi_1\psi}{\pi_1-\pi_0}.$$

- Simple algebra shows that

$$C^{SB} > C^{FB} \Leftrightarrow \pi_0 > 0,$$

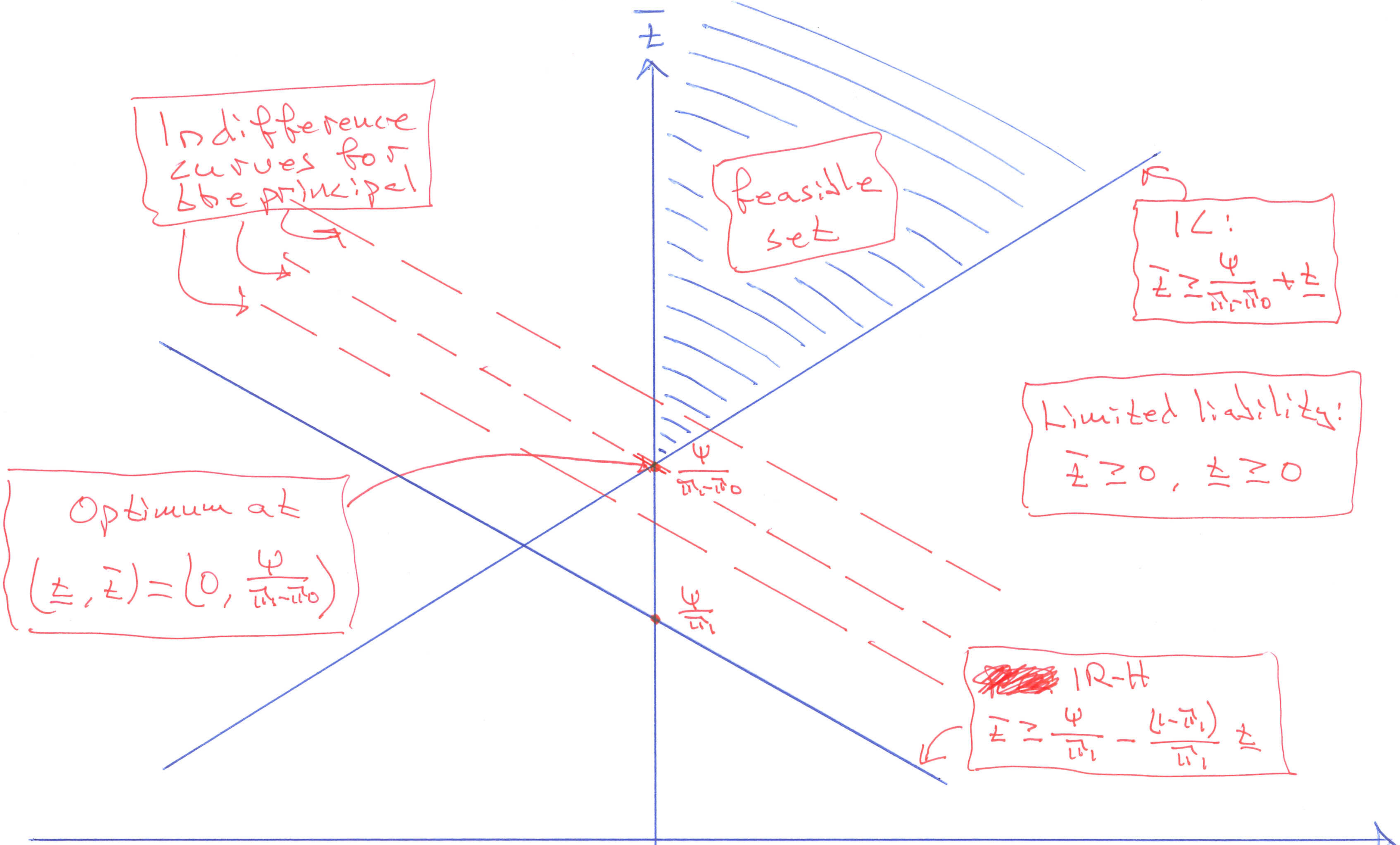
which is satisfied under our assumptions. This means that the cost of implementing the high effort is higher when effort is unobservable compared to when it is observable. Therefore, there will be some parameter values (or, some levels of the benefit of implementing the high effort) for which the high effort is implemented under first best but not under second best — that is the sense in which effort will be underprovided due to asymmetric information.

b) **Relax the assumption that $\widehat{U} = 0$ and allow for any $\widehat{U} \geq 0$. Only consider the case where P wants to induce A to make an effort. Illustrate the second-best solution in a diagram with $\bar{t}$ on the vertical axis and $\underline{t}$ on the horizontal axis. Show in the figure and explain, in qualitative terms, how the nature of the second best solution changes as the outside option utility $\widehat{U}$ becomes larger.**

- For $\overline{U}$ positive and large enough (in particular, for $\overline{U} \geq \frac{\pi_0}{\pi_1 - \pi_0}\psi$) the IR-H constraint becomes binding and the optimal solution is any combination of $\bar{t}$ and $\underline{t}$ such that IR-H binds and both LL-L and IC are satisfied. In terms of a figure (L7, fig 2 — attached at the end of this document), this can be illustrated by moving the graph of IR-H north-east in a parallel fashion until it passes through the original feasible set. The optimal transfer levels are then the ones on the IR-H line and still within the original feasible set.

c) **Suppose that the agent is not protected by limited liability. Explain in words how and why this affects the nature of the second-best solution.**

- In this case the second-best solution will not involve an inefficiency (e.g., it coincides with the first-best solution).
- The economic meaning of the fact that the agent is risk neutral is that he cares only about whether his payment t is large enough *on average*. Hence, the principal can, without violating the individual rationality constraint, incentivize the agent by giving him a negative payment (in practice a penalty) in case of a low output. More generally, the principal can achieve the first-best outcome by making the agent the residual claimant:
  - The agent effectively buys the right to receive any returns ($\bar{q}$ or $\underline{q}$): "the firm is sold to the agent".
  - Thereby, the effort level is chosen by the same individual who bears the consequences of the choice.
  - In this situation the agent makes the same effort choice as the principal would have made.



L7, fig 2

Moral hazard with a risk neutral agent protected by limited liability: